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Two Theorems on Finite Hilbert Transforms*

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Our results are motivated by a theorem of C. Loewner. Let $f(x)$ be a real valued function which is continuously differentiable on an interval (a, b). The function $[f(x) - f(t)]/(x - t)$, extended by continuity to the square $(a, b) \times (a, b)$, is called positive definite if

$$\sum_{j,k=1}^{n} \frac{f(x_k) - f(x_j)}{x_k - x_j} c_j c_k^* \geqslant 0 \tag{1}$$

for all finite sets of points $x_1, \ldots, x_n$ in (a, b) and complex numbers $c_1, \ldots, c_n$. Loewner [7] showed that $[f(x) - f(t)]/(x - t)$ is positive definite if and only if $f(x)$ is the restriction to (a, b) of a function $f(z)$ which is analytic in the upper and lower half-planes and across (a, b) and satisfies $\operatorname{Im} f(z) \geqslant 0$ for $y > 0$. For other treatments and related results see [1, 5, 6, 8, 12]. The first of our two theorems is an extension of Loewner's theorem. The second, a dual result, includes as a special case a characterization of nonnegative integrable functions $g(x)$ on (a, b) such that

$$\int_a^b \left| PV \frac{1}{\pi} \int_a^b \frac{\gamma(t)}{t - x}\, dt \right|^2 g(x)\, dx \leqslant \int_a^b |\gamma(x)|^2 g(x)\, dx$$

for any sufficiently large class of functions $\gamma(x)$.

It is sufficient to treat the case of a symmetric interval $(-a, a)$. By a *locally finite signed measure* (or "*lfs measure*") *on* $(-a, a)$ we mean a real valued set function which is finite and completely additive on the Borel subsets of $(-a, a)$ that are at a positive distance from the end points. The complex conjugate of a number c or function γ is written c^* or γ^*, respectively.

For any $\gamma \in L^1(-\infty, \infty)$ the conjugate function $\tilde{\gamma}$ is defined a.e. on $(-\infty, \infty)$ by

$$\tilde{\gamma}(x) = PV \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\gamma(t)}{t - x}\, dt,$$

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where PV indicates that the integral is taken as a Cauchy principal value. Given $\gamma \in L^1(-a, a)$ we set $\gamma = 0$ off $(-a, a)$ and regard $\tilde{\gamma}$ as defined a.e. on $(-\infty, \infty)$, though we are usually interested only in its restriction to $(-a, a)$. Let $\phi \in L^p(-a, a)$, $\psi \in L^q(-a, a)$ where $p > 1$, $q > 1$. Then $\tilde{\phi} \in L^p(-a, a)$, $\tilde{\psi} \in L^q(-a, a)$. If $p^{-1} + q^{-1} \leqslant 1$ then

$$\int_{-a}^{a} (\phi\tilde{\psi} + \tilde{\phi}\psi)\, dt = 0. \tag{2}$$

If $p^{-1} + q^{-1} < 1$ then [10, p. 169]

$$(\phi\tilde{\psi} + \tilde{\phi}\psi)^{\sim} = \tilde{\phi}\tilde{\psi} - \phi\psi \tag{3}$$

a.e on $(-a, a)$ (and indeed a.e. on $(-\infty, \infty)$).

Let $\mathscr{D}_a$ be the class of continuous functions γ on $(-\infty, \infty)$ which vanish off some closed subinterval of $(-a, a)$ such that $\tilde{\gamma}$ is everywhere defined and continuous on $(-\infty, \infty)$.

PROBLEM 1. *Characterize the class $\mathscr{U}$ of lfs measures μ on $(-a, a)$ such that*

$$\operatorname{Re} \int_{(-a,a)} \tilde{\gamma}\gamma^*\, d\mu \leqslant 0 \tag{4}$$

for all $\gamma \in \mathscr{D}_a$.

We first connect this problem with Loewner's theorem.

PROPOSITION 1. *Let $d\mu = f\,dt$ where f is a real valued function which is integrable over every closed subinterval of $(-a, a)$. Then $\mu \in \mathscr{U}$ if and only if*

$$\lim_{\epsilon \downarrow 0} \int_{-a}^{a} \int_{-a}^{a} \frac{f(x) - f(t)}{x - t}\, \delta_\epsilon(t - x)\, \gamma(t)\, \gamma^*(x)\, dt\, dx \geqslant 0 \tag{5}$$

for all $\gamma \in \mathscr{D}_a$, where $\delta_\epsilon(t) = 1$ or 0 according as $|t| > \epsilon$ or $|t| < \epsilon$. The limit exists for all $\gamma \in \mathscr{D}_a$ whether or not $\mu \in \mathscr{U}$.

If $f(x)$ is continuously differentiable on $(-a, a)$. then (5) is equivalent to (1). Therefore any characterization of the class $\mathscr{U}$ must include Loewner's theorem.

LEMMA 1. *Let γ be continuous on $(-\infty, \infty)$ and vanish off some closed subinterval of $(-a, a)$. If $\tilde{\gamma}$ is equal a.e. to a function which is continuous on $(-\infty, \infty)$ then $\gamma \in \mathscr{D}_a$. In fact, under these conditions*

$$\tilde{\gamma}(x) = \lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{-a}^{a} \frac{\gamma(t)}{t - x}\, \delta_\epsilon(t - x)\, dt$$

uniformly on $(-\infty, \infty)$.

Proof. Let $\tilde{\gamma} = \beta$ a.e. where β is continuous on $(-\infty, \infty)$. Following Titchmarsh [9, pp. 122–125] we define

$$V(x, y) = -\frac{1}{\pi}\int_{-a}^{a} \frac{t-x}{(t-x)^2+y^2}\,\gamma(t)\,dt$$

and argue that

$$V(x, y) = -\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\beta(t)}{(t-x)^2+y^2}\,dt$$

for all real x and $y > 0$. Since β is uniformly continuous on $(-\infty, \infty)$,

$$\beta(x) = -\lim_{y\downarrow 0} V(x, y)$$

uniformly on $(-\infty, \infty)$. Next write

$$\begin{aligned}
V(x, y) &+ \frac{1}{\pi}\int_{-a}^{a} \frac{\gamma(t)}{t-x}\,\delta_y(t-x)\,dt \\
&= V(x, y) + \frac{1}{\pi}\int_{y}^{\infty} \frac{\gamma(x+t)-\gamma(x-t)}{t}\,dt \\
&= -\frac{1}{\pi}\int_{0}^{y} \frac{t}{t^2+y^2}\,[\gamma(x+t)-\gamma(x-t)]\,dt \\
&\quad + \frac{y^2}{\pi}\int_{y}^{\infty} \frac{\gamma(x+t)-\gamma(x-t)}{(t^2+y^2)t}\,dt.
\end{aligned}$$

Let $\epsilon > 0$ be given. Let $|\gamma(t)| \leqslant M$ for all real t, and choose $y_0 > 0$ by uniform continuity so that $|\gamma(x+t) - \gamma(x-t)| < \epsilon$ whenever $0 \leqslant t \leqslant y_0$. Choose y_1 so that $0 < y_1 \leqslant y_0$ and $My_1^2/(\pi y_0^2) < 2\epsilon/3$. Then if only $0 < y < y_1$,

$$\begin{aligned}
&\left| V(x, y) + \frac{1}{\pi}\int_{-a}^{a} \frac{\gamma(t)}{t-x}\,\delta_y(t-x)\,dt \right| \\
&\leqslant \frac{\epsilon}{2\pi y}\int_{0}^{y} dt + \frac{\epsilon y^2}{\pi}\int_{y}^{\infty} \frac{dt}{(t^2+y^2)t} + \frac{2My^2}{\pi}\int_{y_0}^{\infty} \frac{dt}{t^3} \\
&< \epsilon\left(\frac{1}{2\pi} + \frac{y^2}{\pi}\int_{y}^{\infty}\frac{dt}{t^3}\right) + My^2/(\pi y_0^2) \\
&< \epsilon
\end{aligned}$$

uniformly in x.

Proof of Proposition 1. For any $\gamma \in \mathscr{D}_a$,

$$2 \operatorname{Re} \int_{(-a,a)} \tilde{\gamma}\gamma^* \, d\mu = \int_{-a}^{a} \tilde{\gamma}(x)\, \gamma^*(x) f(x)\, dx + \int_{-a}^{a} \tilde{\gamma}^*(t)\, \gamma(t) f(t)\, dt$$

$$= \lim_{\epsilon \downarrow 0} \Big[\int_{-a}^{a} \frac{1}{\pi} \int_{-a}^{a} \frac{\gamma(t)}{t-x} \delta_\epsilon(t-x)\, dt\, \gamma^*(x) f(x)\, dx$$

$$+ \int_{-a}^{a} \frac{1}{\pi} \int_{-a}^{a} \frac{\gamma^*(x)}{x-t} \delta_\epsilon(x-t)\, dx\, \gamma(t) f(t)\, dt \Big]$$

$$= -\lim_{\epsilon \downarrow 0} \frac{1}{\pi} \int_{-a}^{a} \int_{-a}^{a} \frac{f(x)-f(t)}{x-t} \delta_\epsilon(t-x)\, \gamma(t)\, \gamma^*(x)\, dt\, dx.$$

The change of order of integration and passage to the limit is justified by Lemma 1. The result follows.

THEOREM 1. *An lfs measure μ on $(-a, a)$ belongs to the class $\mathscr{U}$ if and only if $d\mu = f\, dx$, where*

$$f(z) = p + \int_{-1/a}^{1/a} \frac{z}{1-zt}\, d\alpha(t), \tag{6}$$

$z \notin (-\infty, -a] \cup [a, \infty)$, *for some real constant p and nondecreasing function $\alpha(t)$ on $[-1/a, 1/a]$.*

Proof of necessity. To begin suppose $a > 1$. The Chebychev polynomials $T_j(x)$, $U_j(x)$ can be defined by

$$T_j(\cos\theta) = \cos j\theta, \qquad U_j(\cos\theta) = \sin(j+1)\theta/\sin\theta$$

for all $j \geqslant 0$. A standard formula [3, p. 254] gives the conjugate pair

$$\gamma_j(x) = \begin{cases} U_j(x)(1-x^2)^{1/2}, & |x| \leqslant 1 \\ 0, & |x| > 1 \end{cases}$$

$$\tilde{\gamma}_j(x) = \begin{cases} -T_{j+1}(x), & |x| \leqslant 1 \\ \text{———}, & |x| > 1 \end{cases}$$

for each $j \geqslant 0$. Also $\gamma_j \in \mathscr{D}_a$, $j \geqslant 0$. For $j = 0$ this is proved by computing $\tilde{\gamma}_0$ on all of $(-\infty, \infty)$ (see [3, p. 246]) and noting that it is continuous. The general case follows from the fact that $\mathscr{D}_a$ is closed under multiplication by x.

Now substitute

$$\begin{cases} \gamma(x) = \sum_{j=0}^{n} c_j U_j(x)(1 - x^2)^{1/2} \chi_{(-1,1)}(x) \\ \tilde{\gamma}(x) \chi_{(-1,1)}(x) = -\sum_{j=0}^{n} c_j T_{j+1}(x) \chi_{(-1,1)}(x) \end{cases} \tag{7}$$

into (4) to get

$$\sum_{j,k=0}^{n} \int_{(-1,1)} [U_k(t) T_{j+1}(t) + U_j(t) T_{k+1}(t)](1 - t^2)^{1/2} d\mu(t) c_j c_k^* \geqslant 0$$

and hence

$$\sum_{j,k=0}^{n} \int_{(-1,1)} U_{j+k+1}(t)(1 - t^2)^{1/2} d\mu(t) c_j c_k^* \geqslant 0$$

for any complex numbers $c_0, ..., c_n$. By the theory of the Hamburger moment problem [11, pp. 129–136] there exists a nondecreasing function $\alpha_0(t)$ such that

$$\int_{(-1,1)} U_{j+1}(t)(1 - t^2)^{1/2} d\mu(t) = \int_{-\infty}^{\infty} t^j d\alpha_0(t)$$

for all $j \geqslant 0$. Since

$$\begin{aligned} \int_{-\infty}^{\infty} t^{2j} d\alpha_0(t) &\leqslant \int_{(-1,1)} |U_{2j+1}(t)| (1 - t^2)^{1/2} d\mu(t) \\ &\leqslant (2j + 2) \int_{(-1,1)} (1 - t^2)^{1/2} d\mu(t), \end{aligned}$$

$\alpha_0(t)$ is constant for $t < -1$ and $t > 1$. If we choose $\alpha_0(t)$ so that $\alpha_0(1) = \alpha_0(1 + 0)$ and $\alpha_0(-1) = \alpha_0(-1 - 0)$ then we can write

$$\int_{(-1,1)} U_{j+1}(t)(1 - t^2)^{1/2} d\mu(t) = \int_{-1}^{1} t^j d\alpha_0(t) \tag{8}$$

for all $j \geqslant 0$.

We show next that μ is absolutely continuous on $(-1, 1)$. By (8) and the identity $U_{j+3}(t) - U_{j+1}(t) = 2T_{j+3}(t)$ we have

$$2 \int_{(-1,1)} T_{j+3}(t)(1 - t^2)^{1/2} d\mu(t) = -\int_{-1}^{1} t^j(1 - t^2) d\alpha_0(t) \tag{9}$$

for all $j \geqslant 0$. Hence for any $n \geqslant 0$,

$$\sum_{j=0}^{n} \left| \int_{(-1,1)} T_{j+3}(t)(1-t^2)^{1/2}\, d\mu(t) \right|^2$$

$$\leqslant \int_{-1}^{1}\int_{-1}^{1} \sum_{j=0}^{n} | st |^j (1-s^2)(1-t^2)\, d\alpha_0(s)\, d\alpha_0(t)$$

$$\leqslant \left(\int_{-1}^{1} d\alpha_0(t)\right)^2.$$

Here we have used the inequality $(1-uv)^{-1}(1-u^2)(1-v^2) < 1$, $u, v \in (0, 1)$. But

$$(1/\pi)^{1/2}\, T_0(t)(1-t^2)^{-1/4}, \qquad (2/\pi)^{1/2}\, T_j(t)(1-t^2)^{-1/4}, \qquad j = 1, 2, 3,...,$$

is a complete orthonormal system in $L^2(-1, 1)$, and so there exists an $h \in L^2(-1, 1)$ such that

$$\int_{(-1,1)} T_j(t)(1-t^2)^{1/2}\, d\mu(t) = \int_{-1}^{1} T_j(t)(1-t^2)^{-1/4}\, h(t)\, dt$$

for all $j \geqslant 0$. Then

$$\int_{(-1,1)} p(t)(1-t^2)^{1/2}\, d\mu(t) = \int_{-1}^{1} p(t)(1-t^2)^{-1/4}\, h(t)\, dt$$

for every polynomial $p(t)$. It follows that, at least on $(-1, 1)$, $d\mu = f\,dt$ where f is real valued and integrable.

We assert that

$$f(x) = \lim_{r\uparrow 1} \frac{2}{\pi} \sum_{j=0}^{\infty} r^j U_j(x) \int_{-1}^{1} U_j(t)(1-t^2)^{1/2} f(t)\, dt$$

a.e. on $(-1, 1)$. This follows from the fact that the Fourier sine series for the integrable function $f(\cos\theta)\sin\theta$, $0 < \theta < \pi$, is Abel summable a.e. to the function [13, p. 99]. Since

$$\frac{1}{1-2xt+t^2} = \sum_{j=0}^{\infty} t^j U_j(x) \tag{10}$$

for $-1 < t < 1$, $-1 < x < 1$, we obtain by (8)

$$\begin{aligned} f(x) &= p_0 + \lim_{r\uparrow 1} \frac{2r}{\pi} \int_{-1}^{1} \sum_{j=0}^{\infty} (rt)^j \, U_{j+1}(x) \, d\alpha_0(t) \\ &= p_0 + \lim_{r\uparrow 1} \frac{2r}{\pi} \int_{-1}^{1} \frac{2x - rt}{1 - 2rxt + r^2t^2} \, d\alpha_0(t) \\ &= p_0 + \frac{2}{\pi} \int_{-1}^{1} \frac{2x - t}{1 - 2xt + t^2} \, d\alpha_0(t) \end{aligned}$$

a.e. on $(-1, 1)$. By redefining f on a set of measure zero we can assume that this representation is valid everywhere on $(-1, 1)$. Since

$$\frac{2x - t}{1 - 2xt + t^2} + \frac{t}{1 + t^2} = 2x \left(1 - \frac{2t}{1 + t^2} \, x\right)^{-1} (1 + t^2)^{-2}$$

we may rewrite the representation in the form

$$f(x) = p_1 + \int_{-1}^{1} \frac{x}{1 - xt} \, d\alpha_1(t), \qquad -1 < x < 1,$$

where p_1 is a real constant and $\alpha_1(t)$ is a nondecreasing function on $[-1, 1]$.

Now let a be an arbitrary positive number, and let b satisfy $0 < b < a$. By making a change of scale and using what was just proved we can argue that μ is absolutely continuous on $(-b, b)$, and on $(-b, b)$ $d\mu = f\,dx$ where

$$f(x) = p_b + \int_{-1/b}^{1/b} \frac{x}{1 - xt} \, d\alpha_b(t), \qquad -b < x < b,$$

for some real constant p_b and nondecreasing function $\alpha_b(t)$ on $[-1/b, 1/b]$. Hence μ is (locally) absolutely continuous on $(-a, a)$. Since

$$p_b = f(0), \qquad \alpha_b(1/b) - \alpha_b(-1/b) = f'(0)$$

a compactness argument shows that $d\mu = f\,dx$ with f of the required form.

Proof of sufficiency. If $\mu_1, \mu_2 \in \mathscr{U}$ and p_1, p_2 are nonnegative numbers, then $p_1\mu_1 + p_2\mu_2 \in \mathscr{U}$. If $(\mu_n)_1^\infty$ is a sequence in $\mathscr{U}$ and if μ is a real valued measure on $(-a, a)$ such that

$$\lim_{n\to\infty} \int_{(-a,a)} \gamma \, d\mu_n = \int_{-a}^{a} \gamma \, d\mu$$

for all continuous functions γ which vanish off some closed subinterval of $(-a, a)$, then $\mu \in \mathscr{U}$. Therefore, it is enough to show that $\mu \in \mathscr{U}$ if $d\mu = f\,dx$

where $f(x) = \pm 1$ or $f(x) = x/(1 - \lambda x)$ for some λ, $-1/a < \lambda < 1/a$. This follows from Proposition 1.

As dual problems we consider the following.

PROBLEM 2. *Determine the class* $\mathscr{V}_+$ *of nonnegative lfs measures* ν *on* $(-a, a)$ *such that*

$$\int_{(-a,a)} |\tilde{\gamma}|^2 \, d\nu \leqslant \int_{(-a,a)} |\gamma|^2 \, d\nu \tag{11}$$

for all $\gamma \in \mathscr{D}_a$.

A similar problem is considered by Helson and Szegö [4].

PROBLEM 3. *Determine the class* $\mathscr{V}$ *of lfs measures* ν *on* $(-a, a)$ *such that*

$$\begin{aligned}\int_{(-b,b)} &|\tilde{\gamma}(t)|^2 (a^2 - t^2)^{1/2} (b^2 - t^2)^{-1/2} \, d\nu(t) \\ &\leqslant \int_{(-b,b)} |\gamma(t)|^2 (a^2 - t^2)^{1/2} (b^2 - t^2)^{-1/2} \, d\nu(t) \end{aligned}\tag{12}$$

for all $\gamma \in \mathscr{D}_a$ *and numbers* b, $0 < b < a$, *such that* γ *vanishes off* $(-b, b)$.

It is not necessary to make any assumption regarding the existence of the integrals in (12). Our understanding of (12) is that the inequality holds for points b such that

$$\int_{(-b,b)} (b^2 - t^2)^{-1/2} \, d\,|\,\nu\,|\,(t) < \infty. \tag{13}$$

The excluded b's form a Lebesgue null set. This follows from Fubini's theorem, since for any ϵ and c, $0 < \epsilon < c < a$, we have

$$\begin{aligned}\int_\epsilon^c &\left[\int_{\epsilon<|t|<x} (x^2 - t^2)^{-1/2} \, d\,|\,\nu\,|\,(t)\right] dx \\ &= \int_{\epsilon<|t|<c} \left[\int_{|t|<x<c} (x^2 - t^2)^{-1/2} \, dx\right] d\,|\,\nu\,|\,(t) \\ &= \int_{\epsilon<|t|<c} \log \frac{c + (c^2 - t^2)^{1/2}}{t} \, d\,|\,\nu\,|\,(t) < \infty.\end{aligned}$$

With this convention it is a consequence that for $\nu \in \mathscr{V}$, (13) holds for all b, $0 < b < a$. See Theorem 2.

PROPOSITION 2. *Let* λ *be an lfs measure on* $(-a, a)$. *Suppose that for some fixed number* b, $0 < b < a$,

$$\int_{(-b,b)} |\tilde{\gamma}|^2 \, d\lambda \leqslant \int_{(-b,b)} |\gamma|^2 \, d\lambda \tag{14}$$

for all γ in $\mathscr{D}_a$ which vanish off $(-b, b)$. Then on $(-b, b)$, $d\lambda = k\,dx$ where

$$k(x) = (b^2 - x^2)^{-1/2} \left[p + qx - \int_{-1/b}^{1/b} \frac{x^2}{1-xt}\, d\eta(t)\right], \tag{15}$$

$-b < x < b$, for some real constants $p = p_b$, $q = q_b$ and nondecreasing function $\eta(t) = \eta_b(t)$ on $[-1/b, 1/b]$.

Proof. The argument follows the proof of necessity of Theorem 1, the main change being that the roles of the polynomials $T_n(x)$, $U_n(x)$ are interchanged.

By making a change of scale we can assume that $a > 1$ and $b = 1$. Substitute (7) into (14) to get

$$\sum_{j,k=0}^{n} \int_{(-1,1)} [U_j(t)\, U_k(t)(1-t^2) - T_{j+1}(t)\, T_{k+1}(t)]\, d\lambda(t)\, c_j c_k^* \geqslant 0$$

and hence

$$-\sum_{j,k=0}^{n} \int_{(-1,1)} T_{j+k+2}(t)\, d\lambda(t)\, c_j c_k^* \geqslant 0$$

for any complex numbers $c_0, \ldots, c_n$. As in the proof of Theorem 1 this implies that

$$-\int_{(-1,1)} T_{j+2}(t)\, d\lambda(t) = \int_{-1}^{1} t^j\, d\eta_0(t) \tag{16}$$

for all $j \geqslant 0$ where $\eta_0(t)$ is a nondecreasing function on $[-1, 1]$. Since $T_{j+2}(t) - T_j(t) = -2(1-t^2)\, U_j(t)$ we obtain

$$-2\int_{(-1,1)} U_{j+2}(t)(1-t^2)\, d\lambda(t) = \int_{-1}^{1} t^j(1-t^2)\, d\eta_0(t)$$

analogous to (9), and from this we argue that λ is absolutely continuous on $(-1, 1)$. Let $d\lambda = k\,dx$ on $(-1, 1)$.

From the Fourier cosine series for $k(\cos\theta)\sin\theta$, $0 < \theta < \pi$, we obtain

$$(1-x^2)^{1/2}\, k(x) = \frac{1}{\pi}\int_{-1}^{1} T_0(t)\, k(t)\, dt + \lim_{r\uparrow 1} \frac{2}{\pi} \sum_{j=1}^{\infty} r^j T_j(x) \int_{-1}^{1} T_j(t)\, k(t)\, dt$$

a.e. on $(-1, 1)$. Also, in place of (10) we have

$$\frac{1-xt}{1-2xt+t^2} = \sum_{j=0}^{\infty} t^j T_j(x) \tag{17}$$

for $-1 < t < 1$, $-1 < x < 1$. Now from (16) we obtain

$$(1 - x^2)^{1/2}\, k(x) = p_0 + q_0 x - \lim_{r\uparrow 1} \frac{2}{\pi} \sum_{j=0}^{\infty} r^{j+2} T_{j+2}(x) \int_{-1}^{1} t^j \, d\eta_0(t)$$

$$= p_0 + q_0 x - \lim_{r\uparrow 1} \frac{2r^2}{\pi} \int_{-1}^{1} \frac{2x^2 - 1 - rtx}{1 - 2rtx + r^2t^2} \, d\eta_0(t)$$

$$= p_0 + q_0 x - \frac{2}{\pi} \int_{-1}^{1} \frac{2x^2 - 1 - tx}{1 - 2tx + t^2} \, d\eta_0(t)$$

where p_0, q_0 are real constants. By redefining $k(x)$ on a set of measure zero we may assume that this representation holds at every point x of $(-1, 1)$. The identity

$$\frac{2x^2 - 1 - tx}{1 - 2tx + t^2} = -\frac{1}{1 + t^2} - \frac{3t + t^3}{(1 + t^2)^2}\, x + \frac{2x^2}{1 - 2tx + t^2} \frac{1 - t^2}{(1 + t^2)^2}$$

is used to rewrite the representation in the form (15).

LEMMA 2. *Let g be a real valued function in $L^p(-b, b)$ where $p > 1$. Let $d\nu = g\, dx$, $d\mu = -\tilde{g}\, dx$ on $(-b, b)$. Then*

$$\int_{(-b,b)} |\, \tilde{\gamma}\, |^2 \, d\nu \leqslant \int_{(-b,b)} |\, \gamma\, |^2 \, d\nu \tag{18}$$

for all $\gamma \in \mathscr{D}_b$ if and only if

$$\operatorname{Re} \int_{(-b,b)} \tilde{\gamma}\gamma^* \, d\mu \leqslant 0 \tag{19}$$

for all $\gamma \in \mathscr{D}_b$. In this case, (18) *and* (19) *hold for any $\gamma \in \mathscr{D}_a$, $a > b$, such that γ vanishes off $(-b, b)$.*

Proof. If $\gamma \in \mathscr{D}_b$ then by (2) and (3),

$$2 \operatorname{Re} \int_{(-b,b)} \tilde{\gamma}\gamma^* \, d\mu = -\int_{-b}^{b} (\tilde{\gamma}\gamma^* + \gamma\tilde{\gamma}^*)\, \tilde{g} \, dt$$

$$= \int_{-b}^{b} (\tilde{\gamma}\gamma^* + \gamma\tilde{\gamma}^*)^{\sim}\, g \, dt$$

$$= \int_{(-b,b)} (|\, \tilde{\gamma}\, |^2 - |\, \gamma\, |^2) \, d\nu.$$

The first assertion is now clear. The second follows by approximation.

PROPOSITION 3. *We have*

$$\int_{-b}^{b} |\tilde{\gamma}(t)|^2 (b^2 - t^2)^{-1/2}\, dt = \int_{-b}^{b} |\gamma(t)|^2 (b^2 - t^2)^{-1/2}\, dt$$

$$\int_{-b}^{b} |\tilde{\gamma}(t)|^2\, t(b^2 - t^2)^{-1/2}\, dt = \int_{-b}^{b} |\gamma(t)|^2\, t(b^2 - t^2)^{-1/2}\, dt$$

for any $\gamma \in \mathscr{D}_b$, *and indeed for any* $\gamma \in \mathscr{D}_a$, $a > b$, *such that* γ *vanishes off* $(-b, b)$.

Proof. The conjugate of $g(x) = (b^2 - x^2)^{-1/2}$, $x \in (-b, b)$, is 0 in $(-b, b)$ by [3, p. 247]. The conjugate of $g(x) = x(b^2 - x^2)^{-1/2}$, $x \in (-b, b)$, is 1 in $(-b, b)$. In either case (19) holds for $d\mu = \pm\tilde{g}\, dx$ and any $\gamma \in \mathscr{D}_b$. By Lemma 2, (18) holds for $d\nu = \pm g\, dx$ and any $\gamma \in \mathscr{D}_a$, $a > b$, such that γ vanishes off $(-b, b)$. The result follows.

LEMMA 3. *If* $\lambda \in [-1/a, 1/a]$ *and* $0 < b < a$, *then*

$$-\int_{-b}^{b} |\tilde{\gamma}(t)|^2\, t^2(b^2 - t^2)^{-1/2} (1 - \lambda t)^{-1}\, dt$$
$$\leqslant -\int_{-b}^{b} |\gamma(t)|^2\, t^2(b^2 - t^2)^{-1/2} (1 - \lambda t)^{-1}\, dt$$

for any $\gamma \in \mathscr{D}_a$ *such that* γ *vanishes off* $(-b, b)$.

Proof. We compute the conjugate of $g(x) = -x^2(b^2 - x^2)^{-1/2}(1 - \lambda x)^{-1}$, $x \in (-b, b)$. By [3, p. 247]

$$PV \frac{1}{\pi} \int_{-b}^{b} \frac{1}{t - x} (b^2 - t^2)^{-1/2}\, dt = \begin{cases} 0, & |x| < b \\ -\operatorname{sgn} x (x^2 - b^2)^{-1/2}, & |x| > b \end{cases} \tag{20}$$

Hence if $-b < x < b$,

$$\begin{aligned} &PV \frac{1}{\pi} \int_{-b}^{b} \frac{1}{t - x} (b^2 - t^2)^{-1/2} (1 - \lambda t)^{-1}\, dt \\ &\quad = (1 - \lambda x)^{-1}\, PV \frac{1}{\pi} \int_{-b}^{b} \left[\frac{1}{t - x} - \frac{1}{t - 1/\lambda}\right] (b^2 - t^2)^{-1/2}\, dt \\ &\quad = (1 - \lambda x)^{-1} \operatorname{sgn} \lambda (\lambda^{-2} - b^2)^{-1/2} \\ &\quad = \lambda(1 - b^2\lambda^2)^{-1/2}/(1 - \lambda x) \end{aligned}$$

at least for $\lambda \neq 0$. By (20) the same formula holds for $\lambda = 0$ as well. If $\lambda \neq 0$ then

$$g(x) = (\lambda^{-2} + \lambda^{-1}x)(b^2 - x^2)^{-1/2} - \lambda^{-2}(b^2 - x^2)^{-1/2}/(1 - \lambda x),$$

$x \in (-b, b)$, and hence

$$-\tilde{g}(x) = -\lambda^{-1} + \lambda^{-1}(1 - b^2\lambda^2)^{-1/2}/(1 - \lambda x),$$

$x \in (-b, b)$. By Proposition 1, (19) holds for $d\mu = -\tilde{g}\, dx$ and any $\gamma \in \mathscr{D}_b$. It holds when $\lambda = 0$ by a separate argument. The result now follows from Lemma 2.

THEOREM 2. *An lfs measure ν on $(-a, a)$ belongs to the class $\mathscr{V}$ if and only if $d\nu = g\, dx$ where*

$$g(z) = (a^2 - z^2)^{-1/2} \left[p + qz - \int_{-1/a}^{1/a} \frac{z^2}{1 - zt}\, d\beta(t)\right], \tag{21}$$

$z \notin (-\infty, -a] \cup [a, \infty)$, for some real constants p, q and nondecreasing function $\beta(t)$ on $[-1/a, 1/a]$.

Proof. Suppose $\nu \in \mathscr{V}$. Let b be any number in $(0, a)$ such that (13) holds. Let λ be any real valued measure on $(-a, a)$ such that on $(-b, b)$, $d\lambda = (a^2 - t^2)^{1/2}(b^2 - t^2)^{-1/2}\, d\nu$. By Proposition 2, $d\lambda = k\, dx$ on $(-b, b)$ where $k(x)$ has the form (15). Since the set of points b in $(0, a)$ such that (13) holds has a as a limit point, it follows that $d\nu = g\, dx$ on $(-a, a)$. Further, for points b arbitrarily near a,

$$g(x) = (a^2 - x^2)^{-1/2} \left[p_b + q_b x - \int_{-1/b}^{1/b} \frac{x^2}{1 - xt}\, d\beta_b(t)\right], \qquad -b < x < b,$$

where p_b, q_b are real constants and $\beta_b(t)$ is a non-decreasing function on $[-1/b, 1/b]$. A straightforward compactness argument shows that g can be written in the form (21).

The sufficiency follows from Proposition 3 and Lemma 3.

PROPOSITION 4. *An lfs measure ν on $(-a, a)$ belongs to the class $\mathscr{V}_+$ if and only if $d\nu = g\, dx$ where g is a nonnegative function on $(-a, a)$ of the form*

$$g(x) = (a^2 - x^2)^{-1/2} \left[p + qx - \int_{-1/a}^{1/a} \frac{x^2}{1 - xt}\, d\beta(t)\right] \tag{22}$$

where p, q are real constants and $\beta(t)$ is a nondecreasing function on $[-1/a, 1/a]$. Thus $\mathscr{V}_+$ coincides with the class of nonnegative measures in $\mathscr{V}$.

Proof. Suppose $\nu \in \mathscr{V}_+$. Then ν satisfies the hypotheses of Proposition 2 for every $b \in (0, a)$. Hence $d\nu = g\, dx$, and for each fixed b there is a representation

$$g(x) = (b^2 - x^2)^{-1/2} \left[p_b + q_b x - \int_{-1/b}^{1/b} \frac{x^2}{1 - xt}\, d\beta_b(t)\right],$$

$-b < x < b$, where p_b, q_b are real constants and $\beta_b(t)$ is a nondecreasing function on $[-1/b, 1/b]$. By a compactness argument g has a representation in the form (22). Since ν is nonnegative, so is g.

Conversely let $d\nu = g\,dx$ where g is a nonnegative function of the form (22). By Theorem 2, $\nu \in \mathscr{V}$ and hence (12) holds for all $\gamma \in \mathscr{D}_a$ and all $b \in (0, a)$ such that γ vanishes off $(-b, b)$. Since ν is nonnegative we may let $b \uparrow a$ and apply Fatou's lemma to get (11). That is, $\nu \in \mathscr{V}_+$.

Note added in proof. In connection with the Loewner problem, see also W. F. Donoghue, Jr., "Monotone Matrix Functions and Analytic Continuation," Springer Verlag, New York-Heidelberg-Berlin, 1974.

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